Nonstationary queues: Estimation of the rate of convergence

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Abstract. The paper is devoted to the estimation of the rate of of exponential convergence of nonhomogeneous queues exhibiting different types of ergodicity. The main tool of our study is the method, which was proposed by the second author in the late 1980-s and was subsequently extended and developed in different directions in a series of joint papers by the authors of the present paper. The method originated from the idea of Gnedenko and Makarov to employ the logarithmic norm of a matrix to the study of the problem of stability of nonhomogeneous Markov chains. In the present paper we apply the method to a class of Markov queues with a special form of nonhomogenuity that is common in applications.

1 Introduction.

It is commonly acknowledged that explicit expressions for the transient behaviour of functionals of stochastic models can be found only in a few special cases. In view of this, efforts of generations of probabilists have been focused on the study of the rate of convergence, as time $t \to \infty$, to the steady state of a process. During the last two decades a remarkable progress was made in this field, regarding time-homogeneous Markov chains, as a result of implementation and development of sophisticated techniques: coupling, logarithmic Sobolev inequalities, the Poincaré inequality and its versions arising from the variational interpretation of eigenvalues, and duality. For reference, we recommend recent review and research papers [1], [3], [10], [12], [8], [4], [11]. The new stream of nowadays research was motivated by new fields of applications, s.t. algorithms of Monte Carlo for simulation of Markov chains and enumeration algorithms in computer science and group theory.

A variety of problems for queueing models, among them stability, were investigated by V. Kalashnikov [22].

There is also a growing interest in time-nonhomogeneous Markov chains (see [14], [21], [23]) that model a large number of real queuing systems. Most of research on nonhomogeneous queues is devoted to various methods of approximation of their transient behaviour(see [7], [27], [28] and references therein.) A quite different approach is used in the study of annealing processes (for references see [6]). Here Markov chains considered have exponentially vanishing intensities, as $t \to \infty$. The research in this field is aimed to estimate the rate of convergence of the process to the invariant distribution.

Our paper is devoted to the estimation of the rate of of exponential convergence of nonhomogeneous queues exhibiting different types of ergodicity. The main tool of our study is the method, which was proposed by the second author in the late 1980s (see [37]) and was subsequently extended and developed in different directions in a series of joint papers by the authors of the present paper. The method originated from the idea of Gnedenko and Makarov([15]) to employ the logarithmic norm of a matrix to the study of the problem of stability of Kolmogorov system of differential equations associated with nonhomogeneous Markov chains. The method is based on the following two ingredients: the logarithmic norm of a linear operator and a special similarity transformation of the matrix of intensities of the Markov chain considered. In the paper we apply the method to a new general class of Markov queues with a special form of nonhomogenuity that is common in applications. Namely, we consider the case of asymptotically periodic arrival and service rates.

The present paper is a substantially modified and extended version of our preliminary report [19].

2 Preliminaries.

We consider a contionuous time nonhomogeneous birth and death process (BDP) $X(t),\ t \geq 0$ on the state space $E = \{0, 1, \ldots, N\},\ N \leq \infty$, with time dependent intensities of birth $\lambda_n(t),\ t \geq 0$, and death $\mu_n(t),\ t \geq 0,\ n \in E$. Namely,

$$P(X(t+h) = j|X(t) = i) = \begin{cases} \lambda_i(t)h + o(h), & if \quad j = i+1 \\ \mu_i(t)h + o(h), & \text{if} \quad j = i-1 \\ 1 - (\lambda_i(t) + \mu_i(t))h + o(h), & \text{if} \quad j = i \\ o(h), & \text{if} \quad |i-j| > 1, \end{cases} \tag{2.1}$$

where $h > 0$ and, for the sake of brevity, $o(h) = o_i(t,h),\ i \in E$ denotes different quantities s.t. for all $t \geq 0$,

$$\lim_{h \to 0} \frac{\sup_i |o_i(t,h)|}{h} = 0.$$

Denote

$$p_{ij}(s,t) = Pr\left(X(t) = j \mid X(s) = i\right),\ i,j \in E, \quad 0 \leq s \leq t \tag{2.2}$$

and $p_i(t) = \Pr(X(t) = i),\ i \in E, \quad t \geq 0$ the transition and the state probabilities respectively.

Let $\mathbf{p} = \mathbf{p}(t) = (p_0(t), \ldots, p_N(t))^T,\ t \geq 0$ be a column vector of probabilities of states and let $A(t), \quad t \geq 0$ be the intensity matrix induced by (2.1).

Then the evolution of the process $X(t)$ is described by the forward Kolmogorov system

$$\frac{d\mathbf{p}}{dt} = A(t)\,\mathbf{p}, \quad \mathbf{p} = \mathbf{p}(t), \quad t \geq 0. \tag{2.3}$$

The Cauchy operator $U(t,s), \quad 0 \leq s \leq t$ of (2.3) is given by the matrix $U^T(t,s) = P(s,t) := (p_{ij}(s,t))_{i,j=0}^N$. We denote throughout the paper by $\|\bullet\|$ the l_1- norm, i. e. $\|\mathbf{x}\| = \sum_{i\in E}|x_i|$, for $\mathbf{x} = (x_0,\dots,x_N)^T$ and $\|B\| = \sup_{j\in E}\sum\limits_{i\in E}|b_{ij}|$, for $B = (b_{ij})_{i,j=0}^N$.

Let Ω be the set of vectors $\Omega = \{\mathbf{x} = (x_0,\dots,x_N)^T : \mathbf{x} \geq 0, \quad \|\mathbf{x}\| = 1\}$. Assuming that $\mathbf{p}(t) \in \Omega$, $t \geq 0$ is a solution of (2.3), we consider the following types of ergodicity of continuous time Markov chains.

Definition 1 . *A Markov chain $X(t), \quad t \geq 0$ on E is called*
ergodic, *if there exists a probability measure $\pi = (\pi(0),\dots,\pi(N))^T$ on E, such that $\lim\limits_{t\to\infty}\|\mathbf{p}(t) - \pi\| = 0$, for all $\mathbf{p} \in \Omega$;*
weakly ergodic, *if $\lim\limits_{t\to\infty}\left\|\mathbf{p}^{(1)}(t) - \mathbf{p}^{(2)}(t)\right\| = 0$, for all $\mathbf{p}^{(i)} \in \Omega$, $i = 1,2$;*
null-ergodic , *if $p_n(t) \to 0, \quad n \in E$, as $t \to \infty$, for all $\mathbf{p} \in \Omega$ and*
quasi-ergodic, *if there exists a probability measure $\tilde{\pi}$ on E, such that*

$$\lim_{t\to\infty}\left\|\frac{1}{t}\int_0^t \mathbf{p}(\tau)\,d\tau - \tilde{\pi}\right\| = 0,$$

for all $\mathbf{p} \in \Omega$.

Remark 1 . *BDP can be both weakly ergodic and null-ergodic, see Example 1 in [19]. Hovewer, quasi-ergodic BDP can not be null-ergodic.*

We deal with the class of nonhomogeneous BDP given by

$$\lambda_n(t) = \lambda_n a(t), \ \mu_n(t) = \mu_n b(t), \quad t \geq 0 \quad n \in E. \tag{2.4}$$

We assume throught the paper that the basic functions $a(t) \geq 0$, $b(t) \geq 0$, $t \geq 0$ are locally integrable on $[0,\infty)$ and asymptotically periodic with periods T_1, T_2 correspondingly. The latter means that

$$a(t) = a_1(t) + a_2(t), \ b(t) = b_1(t) + b_2(t), \quad t \geq 0, \tag{2.5}$$

where

$$a_1(t+T_1) = a_1(t), \ b_1(t+T_2) = b_1(t), t \geq 0, \quad \lim_{t\to\infty}\Big(|a_2(t)| + |b_2(t)|\Big) = 0. \tag{2.6}$$

Regarding the rates λ_n, μ_n, we assume that

$$\lambda_n > 0, \quad n = 0, \dots, N-1, \quad \mu_n > 0, \ n = 1, \dots, N, \ \mu_0 = 0. \tag{2.7}$$

We also assume that in the case $N = \infty$ there exist the limits

$$\lim_{n \to \infty} \lambda_n = \lambda > 0, \ \lim_{n \to \infty} \mu_n = \mu > 0, \tag{2.8}$$

while in the case $N < \infty$,

$$\lambda_N = 0, \quad \lambda := \lambda_{N-1}, \ \mu := \mu_N. \tag{2.9}$$

Some particular cases of the above setting were studied in [17], [25], [16]. In the aforementioned papers only ergodic BDP's were treated. In the present paper we shall investigate the rate of convergence in other cases given by Definition 2.

For reader's convenience, we recall the method of bounding of the the rate of convergence of BDP's that was introduced in [43].

First, we recall the definition of the logarithmic norm that was proposed for finite-dimensional spaces by Lozinskij [26] and generalized to Banach spaces by Daleckij and Krein [5].

Definition 2 *Let $B(t)$, $t \geq 0$ be a one-parameter family of bounded linear operators on a Banach space $\mathcal{B}$ and let I denote the identity operator.*
For a given $t \geq 0$, the number

$$\gamma(B(t)) = \lim_{h \to +0} \frac{\|I + hB(t)\| - 1}{h} \tag{2.10}$$

is called the logarithmic norm of the operator $B(t)$.

If $\mathcal{B}$ is a $(N+1)-$dimensional vector space with l_1- norm, so that the operator $B(t)$ is given by the matrix $B(t) = (b_{ij}(t))_{i,j=0}^{N}, \quad t \geq 0$, then the logarithmic norm of $B(t)$ can be found explicitly:

$$\gamma(B(t)) = \sup_j \left(b_{jj}(t) + \sum_{i \neq j} |b_{ij}(t)| \right), \quad t \geq 0. \tag{2.11}$$

Associate now the family of operators $B(t), \quad t \geq 0$ with the system of differential equations

$$\frac{d\mathbf{x}}{dt} = B(t)\,\mathbf{x}, \quad t \geq 0, \tag{2.12}$$

where the functions $b_{ij}(t),\ 0 \leq i, j \leq N$ are assumed to be locally integrable on $[0, \infty)$. Then the logarithmic norm of the operator $B(t)$ is related to the Cauchy operator $V(t,s),\ 0 \leq s \leq t$ of the system (2.12) :

$$\gamma(B(t)) = \lim_{h \to +0} \frac{\|V(t+h,t)\| - 1}{h}, \quad t \geq 0. \tag{2.13}$$

From the latter one can deduce the following bounds on the $\mathcal{B}$-norm of the Cauchy operator $V(t,s), \quad 0 \leq s \leq t$:

$$e^{-\int_s^t \gamma(-B(\tau))\,d\tau} \leq \|V(t,s)\| \leq e^{\int_s^t \gamma(B(\tau))\,d\tau}, \quad 0 \leq s \leq t. \tag{2.14}$$

Moreover, for any solution $\mathbf{x}(t) \in \mathcal{B}, \quad t \geq 0$ of (2.12) we have

$$\|\mathbf{x}(t)\| \geq e^{-\int_s^t \gamma(-B(\tau))\,d\tau} \|\mathbf{x}(s)\|. \tag{2.15}$$

We will also make use of the fact that if $\mathcal{B} = l_1$ and all non-diagonal elements of B

$$b_{ij}(t) \geq 0, \quad i \neq j, \quad t \geq 0, \tag{2.16}$$

then, by (2.11),

$$\gamma(B(t)) = \sup_j \sum_i b_{ij}(t), \quad t \geq 0, \tag{2.17}$$

and, consequently, for any solution $\mathbf{x}(t),\ t \geq 0$ of (2.12), s.t. $\mathbf{x}(s) \geq \mathbf{0}$, we have

$$\|\mathbf{x}(t)\| \geq e^{\int_s^t \inf_j \sum_i b_{ij}(\tau)\,d\tau} \|\mathbf{x}(s)\|, \quad 0 \leq s \leq t. \tag{2.18}$$

We adopt further on the definition of the decay function $\beta(t)$ of a nonhomogeneous ergodic Markov chain, given by the authors in [17]. (Note that in the case of a homogeneous Markov chain, $\beta(t) \equiv \beta,\ t \geq 0$ is the spectral gap.)

As a result of the previous discussion, we derive the following two-side bounding of the decay function $\beta(t),\ t \geq 0$, provided that the matrix $B(t),\ t \geq 0$ obeys the condition (2.16) :

$$\underline{c}(t) := \inf_j \sum_i b_{ij}(t) \leq -\beta(t) \leq \bar{c}(t) := \sup_j \sum_i b_{ij}(t), \quad t \geq 0. \tag{2.19}$$

In the method considered, the inequalities (2.14), (2.15), (2.19) constitute the main tool for bounding the decay function, as well as the rates of convergence of other types of Markov chains in Definition 1. Finally, we observe that in the case $\underline{c}(t) = \bar{c}(t),\ t \geq 0$, the estimate (2.19) is exact.

A straightforward implementation of (2.19) to the operator $A(t)$ is not effective, since, by the definition of the intensity matrix and (2.11), we have $\gamma\left(A(t)\right) = 0,\ t \geq 0$.

For this reason, we rewrite (2.3) as a system of nonhomogeneous differential equations with a nonsingular matrix. Namely, substituting in (2.3)

$$p_0(t) = 1 - \sum_{i \geq 1} p_i(t), \quad t \geq 0, \tag{2.20}$$

we have

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z}(t) + \mathbf{f}(t), \quad t \geq 0, \tag{2.21}$$

where

$$\begin{aligned} B(t) = \{a_{ij}(t) - a_{i0}(t), \quad i,j = 1, \ldots, N\}, \\ \mathbf{z}(t) = (p_1(t), \ldots, p_N(t))^T, \\ \mathbf{f}(t) = (a_{10}(t), \ldots, a_{N0}(t))^T, \quad t \geq 0. \end{aligned} \tag{2.22}$$

The solution of (2.21) is given by

$$\mathbf{z}(t) = V(t,s)\,\mathbf{z}(s) + \int_s^t V(t,\tau)\,\mathbf{f}(\tau)\, d\tau, \quad 0 \leq s \leq t, \tag{2.23}$$

where $V(t,s)$ is the Cauchy operator of the system (2.21).

The following simple relationship holds between pairs of solutions

$$\mathbf{z}^{(i)} = \mathbf{z}^{(i)}(t), \quad \mathbf{p}^{(i)} = \mathbf{p}^{(i)}(t), \quad t \geq 0, \quad i = 1, 2$$

of (2.21) and (2.3) respectively:

$$\begin{aligned}\left\|\mathbf{p}^{(1)}-\mathbf{p}^{(2)}\right\| &= \left|p_0^{(1)}-p_0^{(2)}\right|+\sum_{i\geq 1}\left|p_i^{(1)}-p_i^{(2)}\right| = \left|1-\sum_{i\geq 1}p_i^{(1)}-\left(1-\sum_{i\geq 1}p_i^{(2)}\right)\right| \\ +\left\|\mathbf{z}^{(1)}-\mathbf{z}^{(2)}\right\| &= \left|\sum_{i\geq 1}\left(p_i^{(1)}-p_i^{(2)}\right)\right|+\left\|\mathbf{z}^{(1)}-\mathbf{z}^{(2)}\right\|, \quad t\geq 0.\end{aligned}$$

Consequently,

$$\left\|\mathbf{z}^{(1)}-\mathbf{z}^{(2)}\right\| \leq \left\|\mathbf{p}^{(1)}-\mathbf{p}^{(2)}\right\| \leq 2\left\|\mathbf{z}^{(1)}-\mathbf{z}^{(2)}\right\|, \quad t\geq 0, \tag{2.24}$$

which will be used in our subsequent study.
At this point, we note that the matrix $B(t), \quad t \geq 0$ in (2.22) lacks the property (2.16) of the original intensity matrix $A(t),\ t \geq 0$ in (2.3). This fact prevents the implementation of (2.19) . To overcome this obstacle, we employ a similarity transformation of $B(t),\ t \geq 0$, that restores the property (2.16). In [18] it was proven the existence of such transformation for general finite homogeneous Markov chains. However, its explicit construction is known so far for homogeneous BDP's only. The required transformation which is in the core of the method considered, was suggested in [38]-[40]. It is given by the upper triangular $N\times N$ matrix

$$D=\begin{pmatrix} d_0 & d_0 & d_0 & \dots \\ 0 & d_1 & d_1 & \dots \\ 0 & 0 & d_2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{2.25}$$

where $d_i>0, \quad i=0,1,\dots,N-1$. We have

$$D^{-1}=\begin{pmatrix} d_0^{-1} & -d_1^{-1} & 0 & \ddots & \\ 0 & d_1^{-1} & -d_2^{-1} & 0 & \ddots \\ \ddots & 0 & \ddots\, d_2^{-1} & \ddots & \ddots \\ & \ddots & \ddots & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Applying this transformation to the matrix $B(t),\ t \geq 0$ in (2.22), leads to the matrix $DB(t)D^{-1},\ t\geq 0$ with the desired property (2.16) :

$$DB(t)D^{-1} = \begin{pmatrix} -(\lambda_0(t)+\mu_1(t)) & d_0 d_1^{-1}\mu_1(t) & 0 & & \cdots \\ d_1 d_0^{-1}\lambda_1(t) & -(\lambda_1(t)+\mu_2(t)) & d_1 d_2^{-1}\mu_2(t) & 0 & \\ 0 & d_2 d_1^{-1}\lambda_2(t) & \ddots & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots \\ & \cdots & \ddots & \ddots & \ddots \end{pmatrix}. \tag{2.26}$$

We set further on, $d_0 = 1$, and $d_N = 0$ for $N < \infty$, and denote $\delta_{i+1} = d_{i+1}/d_i > 0, \quad i = 0, \dots, N-1$. By (2.11), the logarithmic norm of the above matrix equals to

$$\gamma(DB(t)D^{-1}) = -\inf_k \alpha_k(t), \quad t \geq 0, \tag{2.27}$$

where

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \delta_{k+1}\lambda_{k+1}(t) - \delta_k^{-1}\mu_k(t), \quad t \geq 0, \quad k = 0, \dots, N-1. \tag{2.28}$$

(Here and in what follows $\delta_0 = 0$.)
Next, the substitution $\mathbf{z}(t) = D^{-1}\mathbf{y}(t)$, $t \geq 0$, transforms (2.21) into the system with the matrix $DB(t)D^{-1}$, $t \geq 0$. Following [38], [39], we denote

$$\|\mathbf{z}\|_{1D} = \|D\mathbf{z}\|, \quad t \geq 0, \tag{2.29}$$

and $g = \inf_{k>0} d_k$, to obtain, by virtue of (2.25),

$$\|\mathbf{z}\|_{1D} \geq \frac{g}{2}\|\mathbf{z}\|, \quad t \geq 0. \tag{2.30}$$

Now we apply (2.19) to the matrix $DB(t)D^{-1}$, $t \geq 0$, to derive the following effective way of two-side bounding of the decay function $\beta(t)$, $t \geq 0$ of non-homogeneous BDP's:

$$\underline{\alpha}(t) \leq \beta(t) \leq \bar{\alpha}(t), \quad t \geq 0, \tag{2.31}$$

where

$$\underline{\alpha}(t) = \inf_k \alpha_k(t), \quad \bar{\alpha}(t) = \sup_k \alpha_k(t), \quad t \geq 0 \tag{2.32}$$

In this connection note that, by the definition in [17], the decay function $\beta(t)$, $t \geq 0$ should satisfy $\int\limits_0^{+\infty} \beta(t)\,dt = +\infty$, (but is not required to be nonnegative for all $t \geq 0$).

Therefore, (2.31) make sense provided the weights δ_k, $k=0,\ldots,N-1$ in (2.28) are s.t. $\int\limits_0^{+\infty} \underline{\alpha}(t)dt = +\infty$. The way of finding such weights is discussed further on in the present paper.

Next, with the help of (2.14), (2.30) and (2.24) we obtain for weakly ergodic BDP's :

$$\left\|\mathbf{p}^{(1)}(t)-\mathbf{p}^{(2)}(t)\right\|_{1D} \le e^{-\int\limits_s^t \underline{\alpha}(u)du} \left\|\mathbf{p}^{(1)}(s)-\mathbf{p}^{(2)}(s)\right\|_{1D}, \quad 0\le s\le t. \tag{2.33}$$

and

$$\left\|\mathbf{p}^{(1)}(t)-\mathbf{p}^{(2)}(t)\right\| \le \frac{4}{g} e^{-\int\limits_s^t \underline{\alpha}(u)du} \left\|\mathbf{z}^{(1)}(s)-\mathbf{z}^{(2)}(s)\right\|_{1D}, \quad 0\le s\le t. \tag{2.34}$$

Here, in accordance with our notation (2.29) ,

$$\left\|\mathbf{z}^{(1)}(s)-\mathbf{z}^{(2)}(s)\right\|_{1D} \le \sum_{i\ge 1} q_i \left|p_i^{(1)}(s)-p_i^{(2)}(s)\right|, \quad t\ge 0, \tag{2.35}$$

where $q_i=\sum\limits_{m=0}^{i-1} d_m, \quad i=1,\ldots,N.$

Clearly, (2.34), (2.35) are valid also for ergodic BDP's, by taking $\mathbf{p}^{(2)}(t)=\pi, \quad t\ge 0$.

The reasoning analogous to the preceding one enables also to derive a lower bound for $\left\|\mathbf{p}^{(1)}(t)-\mathbf{p}^{(2)}(t)\right\|$. For this purpose, for a sequence of positive weights $\boldsymbol{\delta}=(\delta_1,\ldots,\delta_{N-1})$, define the quantities

$$\zeta_k(t)=\lambda_k(t)+\mu_{k+1}(t)+\delta_{k+1}\lambda_{k+1}(t)+\delta_k^{-1}\mu_k(t), \quad t\ge 0, \quad k=0,\ldots,N-1. \tag{2.36}$$

Denoting

$$\bar{\zeta}(t)=\sup_k \zeta_k(t), \quad t\ge 0. \tag{2.37}$$

we then have, $\gamma(-DB(t)D^{-1})=-\sup_k\zeta_k(t), \quad t\ge 0$, ((2.11) . Consequently, with the help of (2.24) , we obtain

$$\begin{gathered}\left\|\mathbf{p}^{(1)}(t)-\mathbf{p}^{(2)}(t)\right\| \ge \left\|\mathbf{z}^{(1)}(t)-\mathbf{z}^{(2)}(t)\right\| \ge \\ \|D\|^{-1}\left\|D^{-1}\right\|^{-1} e^{-\int\limits_s^t \bar{\zeta}(u)du} \left\|\mathbf{z}^{(1)}(s)-\mathbf{z}^{(2)}(s)\right\|, \quad 0\le s\le t.\end{gathered} \tag{2.38}$$

Now we are in a position to describe the organization of the paper and its main results. As we already mentioned, we focus our study on a class of nonhomogeneous BDP's with intensities (2.4)-(2.6) . Our main effort is devoted to estimation of $\underline{\alpha}(t),\ t \geq 0$ for these BDP's. This is done in Section 3 and 4, by analyzing the specific form of the expressions (2.28) in the case considered. As a result, we derive estimates of the rate of exponential convergence in different types of ergodicity. As a by-product of this study, we obtain also estimates of important for applications characteristics of transient behaviour of the above processes. The rest four sections are devoted to four particular nonhomogeneous queues widely known in the literature. We achieve an improvement of known estimates of the rate of convergence. We also obtain estimates of the expected length of queues. In the conclusion note that the stability properties of BDP's were explored in [37], [41], [19] and [44].

3 Quasi-ergodic BDP

Denote

$$f_k = \frac{\Delta\lambda\mu_{k+1} - \lambda_{k-1}\mu + \sqrt{\left(\Delta\lambda\mu_{k+1} - \lambda_{k-1}\mu\right)^2 + 4\Delta\lambda\mu\left(\lambda_{k-1}\mu_{k+1} - \lambda_k\mu_k\right)}}{2\Delta\lambda\mu}, \quad k = 0, \dots, N-1, \tag{3.39}$$

where $\Delta > 0$ is a given number, and λ, μ are defined in (2.8), (2.9) .

We denote $\tilde{E} = \{0, 1, \dots, N-1\}$, if $N < \infty$ and $\tilde{E} = E$, if $N = \infty$ and set in (2.5),(2.6)

$$a_m := \lim_{t\to\infty} \frac{1}{t}\int_0^t a(u)\,du = \frac{1}{T_1}\int_0^{T_1} a_1(u)\,du; \quad b_m := \lim_{t\to\infty}\frac{1}{t}\int_0^t b(u)\,du = \frac{1}{T_2}\int_0^{T_2} b_1(u)\,du. \tag{3.40}$$

Theorem 1 *Let for some $\Delta > 1$ the following conditions fulfilled:*

a) for any k $\quad \lambda_{k-1}\mu_{k+1} - \lambda_k\mu_k \geq 0;$

b)

$$\mu b_m - \Delta\lambda a_m > 0, \tag{3.41}$$

c)

$$\inf_{k\in\tilde{E}} f_k := f > 0. \tag{3.42}$$

Let $c \in (0;1)$ be arbitrary number such that a) $c < \frac{\mu_{k+1}}{\mu}, k \geq 0$ and b) $c \leq f$. Then there exists a sequence $\boldsymbol{\delta} = (\delta_k, \quad k \in \tilde{E}) > \mathbf{0}$, such that $\liminf \delta_k > 1, \quad if \quad N = \infty$ and

(i)

$$\underline{\alpha}(t) \geq l(t) := c\left(\mu b(t) - \Delta\lambda a(t)\right), \quad t \geq 0, \tag{3.43}$$

(ii) BDP $X(t), \quad t \geq 0$ is weakly ergodic, so that

$$\left\|\mathbf{p}^{(1)}(t) - \mathbf{p}^{(2)}(t)\right\|_{1D} \leq e^{-\int\limits_s^t l(u)du} \left\|\mathbf{p}^{(1)}(s) - \mathbf{p}^{(2)}(s)\right\|_{1D}, \quad 0 \leq s \leq t, \tag{3.44}$$

and

$$\left\|\mathbf{p}^{(1)}(t) - \mathbf{p}^{(2)}(t)\right\| \leq \frac{4}{g} e^{-\int\limits_s^t l(u)du} \sum_{i \geq 1} q_i \left|p_i^{(1)}(s) - p_i^{(2)}(s)\right|, \quad 0 \leq s \leq t, \tag{3.45}$$

where g and q_i are defined in $(2.30), (2.34)$,

(iii) BDP $X(t), \quad t \geq 0$ is quasi-ergodic.

Proof. The condition $0 < c \leq f \leq f_k$ implies the inequality

$$c^2\Delta\lambda\mu - c\left(\Delta\lambda\mu_{k+1} - \lambda_{k-1}\mu\right) + \lambda_k\mu_k - \lambda_{k-1}\mu_{k+1} \leq 0, \quad k \in \tilde{E} \tag{3.46}$$

for all k. Therefore using the inequality $c < \frac{\mu_{k+1}}{\mu}$, we obtain

$$\frac{\mu_k}{\mu_{k+1} - c\mu} \leq \frac{\lambda_{k-1} + c\Delta\lambda}{\lambda_k}, \quad k \in \tilde{E}. \tag{3.47}$$

Put now

$$\delta_k \in \left[\frac{\mu_k}{\mu_{k+1} - c\mu}, \frac{\lambda_{k-1} + c\Delta\lambda}{\lambda_k}\right], \quad k \in \tilde{E}. \tag{3.48}$$

Then

$$\delta_{k+1}\lambda_{k+1} - \lambda_k \leq c\Delta\lambda, \quad \mu_{k+1} - \delta_k^{-1}\mu_k \geq c\mu, \quad k \in \tilde{E}, \tag{3.49}$$

and, finally

$$\alpha_k(t) \geq c\left(\mu b(t) - \Delta\lambda a(t)\right), \quad k \in \tilde{E}, \quad t \geq 0, \tag{3.50}$$

because in the case considered

$$\alpha_k(t) = b(t)\left(\mu_{k+1} - \delta_k^{-1}\mu_k\right) - a(t)\left(\delta_{k+1}\lambda_{k+1} - \lambda_k\right), \quad k \in \tilde{E}, \quad t \geq 0. \tag{3.51}$$

Now, for the case $N = \infty$ we get from (3.48) as $k \to \infty$

$$\liminf \delta_k \geq \frac{\mu}{\mu - c\mu} = \frac{1}{1-c} > 1$$

because $c < 1$.

Corollary 1 . *If under the assumptions of Theorem 1, the BDP $X(t), \quad t \geq 0$ considered has a stationary distribution π on E, then the BDP is ergodic and*

$$\|\mathbf{p}(t) - \pi\| \leq \frac{4}{g} e^{-\int_s^t l(u)du} \sum_{i \geq 1} q_i \left|p_i(s) - \pi_i\right|, \quad 0 \leq s \leq t. \tag{3.52}$$

Corollary 2 . *Under the assumptions of Theorem 1, the BDP $X(t), \quad t \geq 0$ considered is exponentially weakly ergodic and for any $\varepsilon > 0$ there exists a constant K such that*

$$exp\left(-\int_s^t l(u)\,du\right) \leq Ke^{-(l-\varepsilon)(t-s)}, \quad t \geq 0, \tag{3.53}$$

where $l = c\left(\mu b_m - \Delta\lambda a_m\right).$

Theorem 2 *Under the assumptions of Theorem 1, for any $\varepsilon > 0$ there exists a constant K such that*

$$\Pr\left(X(t) \leq j \mid X(0) = 0\right) \geq 1 - q_{j+1}^{-1} K\lambda_0 a_m \left(1 + \frac{e^{(l-\varepsilon)T_1}}{e^{(l-\varepsilon)T_1} - 1}\right). \tag{3.54}$$

Proof. Put in (2.21) $\mathbf{y}(t) = D\mathbf{z}(t)$, then we have

$$\frac{d\mathbf{y}}{dt} = DB(t)\,D^{-1}\mathbf{y} + D\mathbf{f}(t), \quad t \geq 0. \tag{3.55}$$

The matrix $DB(t)\,D^{-1}$ satisfies (2.16), so under the conditions $\mathbf{y}(0) = \mathbf{0}, \quad \mathbf{f}(t) \geq \mathbf{0}, \quad t \geq 0$, we get $\mathbf{y}(t) \geq \mathbf{0}, \quad t \geq 0$. Let $V_1(t,s)$ be a Cauchy matrix for the equation (3.55). Then $\|V_1(t,s)\| = \|V(t,s)\|_{1D}$. Finally, applying (3.53) we get the following bound in l_{1D} norm

$$\begin{aligned}\sum_{i\geq 1} q_i p_i(t) = \|\mathbf{y}(t)\| \leq \|V_1(t,0)\|\,\|\mathbf{y}(0)\| + \int_0^t \|V_1(t,\tau)\|\,\|D\mathbf{f}(\tau)\|\,d\tau \\ = \int_0^t \|V_1(t,\tau)\|\,\|D\mathbf{f}(\tau)\|\,d\tau \leq K\lambda_0 e^{(-(l-\varepsilon)t}\int_0^t e^{((l-\varepsilon)\tau)}a(\tau)\,d\tau \\ \leq K\lambda_0 a_m\left(1 + \frac{e^{(l-\varepsilon)T_1}}{e^{(l-\varepsilon)T_1}-1}\right).\end{aligned} \tag{3.56}$$

Then

$$\sum_{i>j} p_i(t) \leq q_{j+1}^{-1}\sum_{i>j} q_i p_i(t) \leq$$
$$q_{j+1}^{-1}K\lambda_0 a_m\left(1 + \frac{e^{(l-\varepsilon)T_1}}{e^{(l-\varepsilon)T_1}-1}\right).$$

Let $E(t;k) = \sum_{i\geq 1} i p_i(t), \quad t \geq 0, k \geq 0$ be the mean of the process at time t, if $X(0) = k$.

Corollary 3 . *Under the assumptions of Theorem 1,*

$$E(t;k) \leq \frac{1}{\mathrm{W}}K\lambda_0 a_m\left(1 + \frac{e^{(l-\varepsilon)T_1}}{e^{(l-\varepsilon)T_1}-1}\right), \forall\epsilon > 0, \tag{3.57}$$

where $\mathrm{W} = \inf_{i\geq 1}\frac{q_i}{i} > 0$ *and* l *is defined as in (3.53)*

Proof. The claim follows from the inequality

$$\sum_{i\geq 1} i p_i(t) = \sum_{i\geq 1}\frac{i}{q_i}q_i p_i(t) \leq \frac{1}{\mathrm{W}}\sum_{i\geq 1} q_i p_i(t), \quad t \geq 0.$$

Consider now the case of finite state space. In this case the matrix D in (2.25) is also finite and we can obtain two-sided bounds both for the decay function and for $\left\|\mathbf{p}^{(1)}(t) - \mathbf{p}^{(2)}(t)\right\|, \quad t \geq 0$.

Theorem 3 *Let the conditions of Theorem 1 hold and, in addition, $N<\infty$.*

Then

(i) for any $s\geq 0,\ t\geq s$, and any $\mathbf{p}^1(s)\in\Omega,\ \mathbf{p}^2(s)\in\Omega$

$$e^{-\int\limits_s^t \bar{\zeta}(u)du}\left\|\mathbf{p}^1(s)-\mathbf{p}^2(s)\right\|_{1D}\leq\left\|\mathbf{p}^{(1)}(t)-\mathbf{p}^{(2)}(t)\right\|_{1D}$$
$$\leq e^{-\int\limits_s^t l(u)du}\left\|\mathbf{p}^{(1)}(s)-\mathbf{p}^{(2)}(s)\right\|_{1D},\quad 0\leq s\leq t, \tag{3.58}$$

and

$$\frac{g}{4NG}e^{-\int\limits_s^t \bar{\zeta}(u)du}\left\|\mathbf{p}^1(s)-\mathbf{p}^2(s)\right\|\leq\left\|\mathbf{p}^1(t)-\mathbf{p}^2(t)\right\|$$
$$\leq\frac{4NG}{g}e^{-\int\limits_s^t \underline{\alpha}(u)du}\left\|\mathbf{p}^1(s)-\mathbf{p}^2(s)\right\| \tag{3.59}$$

where $g=\min d_k,\ G=\max d_k$, and $\underline{\alpha}(t),\quad \bar{\zeta}(t),\quad t\geq 0$ are defined in (2.37) and (2.32) respectively;

(ii) for any $s\geq 0,\ t\geq s$, and $\mathbf{p}^1(s)\leq\mathbf{p}^2(s)$

$$\left\|\mathbf{p}^1(t)-\mathbf{p}^2(t)\right\|_{1D}\geq e^{-\int\limits_s^t \bar{\alpha}(u)du}\left\|\mathbf{p}^1(s)-\mathbf{p}^2(s)\right\|_{1D}, \tag{3.60}$$

and

$$\left\|\mathbf{p}^1(t)-\mathbf{p}^2(t)\right\|\geq\frac{g}{4NG}e^{-\int\limits_s^t \bar{\alpha}(u)du}\left\|\mathbf{p}^1(s)-\mathbf{p}^2(s)\right\|. \tag{3.61}$$

Proof. We have to prove only estimates in l_1 norm. We have

$$\|D\|=\sum_{i=0}^{N-1}d_i\leq NG;\quad \left\|D^{-1}\right\|=2\max d_k^{-1}\leq\frac{2}{g}. \tag{3.62}$$

Then

$$\begin{aligned}
&\left\|\mathbf{p}^1(t)-\mathbf{p}^2(t)\right\|\leq 2\left\|\mathbf{z}^1(t)-\mathbf{z}^2(t)\right\|\leq\\
&2\|D\|\left\|D^{-1}\right\|e^{-\int\limits_s^t \underline{\alpha}(u)du}\left\|\mathbf{z}^1(s)-\mathbf{z}^2(s)\right\|\\
\leq\ &\frac{4NG}{g}e^{-\int\limits_s^t \underline{\alpha}(u)du}\left\|\mathbf{z}^1(s)-\mathbf{z}^2(s)\right\|\leq\\
&\frac{4NG}{g}e^{-\int\limits_s^t \underline{\alpha}(u)du}\left\|\mathbf{p}^1(s)-\mathbf{p}^2(s)\right\|.
\end{aligned} \tag{3.63}$$

On the other hand inequality (2.15) implies the bound

$$\begin{aligned} & \left\|\mathbf{p}^1(t)-\mathbf{p}^2(t)\right\| \geq \left\|\mathbf{z}^1(t)-\mathbf{z}^2(t)\right\| \geq \\ & \|D\|^{-1}\left\|D^{-1}\right\|^{-1} e^{-\int_s^t \bar{\zeta}(u)du}\left\|\mathbf{z}^1(s)-\mathbf{z}^2(s)\right\| \\ \geq\ & \frac{g}{2NG} e^{-\int_s^t \bar{\zeta}(u)du}\left\|\mathbf{z}^1(s)-\mathbf{z}^2(s)\right\| \geq \\ & \frac{g}{4NG} e^{-\int_s^t \bar{\zeta}(u)du}\left\|\mathbf{p}^1(s)-\mathbf{p}^2(s)\right\|. \end{aligned} \tag{3.64}$$

Remark 2 . *The rate of convergence in the previous theorems is exponential. Therefore, our bounds provide an approach for obtaining integral estimates for the state probabilities and for the mean. Such estimates were obtained in [4], [34].*

4 Null-ergodic BDPs

Put

$$\alpha_k^{(0)}(t) = \lambda_k(t) + \mu_k(t) - \delta_{k+1}\lambda_k(t) - \delta_k^{-1}\mu_k(t), \quad k = 0, \dots, N-1, \quad t \geq 0. \tag{4.65}$$

and

$$\underline{\alpha}^{(0)}(t) = \inf_k \alpha_k^{(0)}(t). \tag{4.66}$$

Let D be a diagonal matrix

$$\mathrm{D} = diag\{d_0, d_{1,\dots}\}, \tag{4.67}$$

where $d_0 = 1, \quad d_i > 0, \quad i \in E$.
We define the induced $l_{1\mathrm{D}}-$ norm by

$$\|\mathbf{x}\|_{1\mathrm{D}} = \|\mathrm{D}\mathbf{x}\|_1 = \sum_{i\geq 0} d_i |x_i|, \quad \mathbf{x} = (x_1, \dots, \dots) \in l_1.$$

In the case considered all results are based on the bounds for the logarithmic norm $\gamma(A(t))_{1\mathrm{D}} = \gamma(\mathrm{D}A(t)\mathrm{D}^{-1})_1$, where $A(t), \quad t \geq 0$ is defined in (2.3).

Hence,

$$\mathrm{D}A(t)\mathrm{D}^{-1} = \begin{pmatrix} -\lambda_0(t) & d_0 d_1^{-1}\mu_1(t) & 0 & \dots & \\ d_1 d_0^{-1}\lambda_0(t) & -(\lambda_1(t)+\mu_1(t)) & d_1 d_2^{-1}\mu_2(t) & 0 & \\ 0 & d_2 d_1^{-1}\lambda_1(t) & \ddots & \ddots & \ddots \\ \vdots & 0 & \ddots & \ddots & \ddots \\ & \dots & \ddots & \ddots & \ddots \end{pmatrix}, \quad t \geq 0. \tag{4.68}$$

Put

$$h_k = \frac{\Delta\mu\lambda_{k-1} - \lambda\mu_k}{\Delta\lambda\mu}, \quad k \in \tilde{E}. \tag{4.69}$$

Theorem 4 *Let for some* $\Delta > 1$

$$\inf_{k\in\tilde{E}} h_k = h > 0, \tag{4.70}$$

let c *be a positive number under the conditions:*

$$c < 1; \;\; c < \frac{\lambda_k}{\lambda}, k \geq 0, \;\; c \leq h, \tag{4.71}$$

and let

$$\lambda a_m - \Delta\mu b_m > 0. \tag{4.72}$$

Then there exists a sequence $\boldsymbol{\delta} = (\delta_k, \quad k \in \tilde{E}) > \mathbf{0}$*, such that*
(i)

$$\underline{\alpha}^{(0)}(t) \geq \theta(t) = c\left(\lambda a(t) - \Delta\mu b(t)\right), \tag{4.73}$$

where $\liminf \delta_k < 1, \quad if \quad N = \infty$;
(ii) BDP $X(t), \quad t \geq 0$ *is null-ergodic and*

$$\sum_{i=0}^{\infty} d_i p_i(t) \leq G e^{-\int_s^t \theta(u)du}, \quad 0 \leq s \leq t, \tag{4.74}$$

where $G = \sup d_k < \infty$.

Proof.

First we prove the inequalities

$$\alpha_k^{(0)}(t) = a(t)\lambda_k(1-\delta_{k+1}) - b(t)\mu_k\left(\delta_k^{-1}-1\right) \geq c\left(\lambda a(t) - \Delta\mu b(t)\right), \tag{4.75}$$

$k \in \tilde{E}, \quad t \geq 0.$
It is sufficient to verify that

$$\lambda_k \left(1 - \delta_{k+1}\right) \geq c\lambda, \quad \mu_k \left(\delta_k^{-1} - 1\right) \leq c\mu\Delta, \quad k \in \tilde{E}. \tag{4.76}$$

and (4.76) follows from

$$\delta_k \leq \frac{\lambda_{k-1} - c\lambda}{\lambda_{k-1}}, \tag{4.77}$$

$$\delta_k \geq \frac{\mu_k}{\mu_{k+1} + c\mu\Delta}, \quad k \in \tilde{E}. \tag{4.78}$$

If $N = \infty$ and $0 < c < 1$, we have from (4.77)

$$\limsup \delta_k \leq \frac{\lambda - c\lambda}{\lambda} = 1 - c < 1. \tag{4.79}$$

Hence, (4.77), (4.78) imply

$$\frac{\mu_k}{\mu_{k+1} + c\mu\Delta} \leq \frac{\lambda_{k-1} - c\lambda}{\lambda_{k-1}}, \quad k \in \tilde{E} \tag{4.80}$$

Since $0 < c < \frac{\lambda_k}{\lambda}$, the latter is equivalent to

$$c^2 \Delta\lambda\mu - c\left(\Delta\lambda_{k-1}\mu - \lambda\mu_k\right) \leq 0, \quad k \in \tilde{E}. \tag{4.81}$$

For fixed k the largest root of (4.81) is h_k given by (4.69).
So, we have

$$\gamma\left(A(t)\right)_{1\mathrm{D}} \leq -\theta(t) = -c\left(\lambda a(t) - \Delta\mu b(t)\right), \quad t \geq 0 \tag{4.82}$$

Then

$$\sum_{i=0}^{\infty} d_i p_i(t) = \|\mathbf{p}(t)\|_{1\mathrm{D}} \leq e^{-\int_s^t \theta(u)du} \|\mathbf{p}(s)\|_{1\mathrm{D}} \leq G e^{-\int_s^t \theta(u)du}, \quad 0 \leq s \leq t, \tag{4.83}$$

for any $\mathbf{p}(s) \in \Omega$. This implies the exponential null-ergodicity of BDP.

Now we are in position to bound the state probabilities in null-ergodic case. Put $d_j^{\min} = \min_{i \leq j} d_i > 0, \quad j \in E.$

Corollary 4 . *Under assumptions of Theorem 4,*

$$p_k(t) \le \frac{G}{d_k} e^{-\int\limits_s^t \theta(u)\,du}, \quad k \in E, \quad 0 \le s \le t. \tag{4.84}$$

and

$$\Pr\left(X(t) \le j \mid X(0) = k\right) \le \frac{d_k}{d_j^{\min}} e^{-\int\limits_0^t \theta(u)\,du}, \quad j, k \in E, \quad t \ge 0 \tag{4.85}$$

Proof. We have

$$p_k(t) \le d_k^{-1} \sum_{i=0}^{\infty} d_i p_i(t) \le \frac{G}{d_k} e^{-\int\limits_s^t \theta(u)\,du}. \tag{4.86}$$

Thus,

$$\begin{aligned} d_j^{\min} \sum_{i=0}^{j} p_i(t) &\le \sum_{i=0}^{j} d_i p_i(t) \le \sum_{i=0}^{\infty} d_i p_i(t) \\ &\le e^{-\int\limits_0^t \theta(u)\,du} \left\|\mathbf{p}(0)\right\|_{1D} = d_k e^{-\int\limits_0^t \theta(u)\,du}, \end{aligned} \tag{4.87}$$

which implies (4.85).

Remark 3 *If BDP is null-ergodic then* $\mathrm{E}(t;k) \to \infty$ *as* $t \to \infty$ *for any fixed* $k \in E$. *Put*

$$r(t) = \min\left\{\lambda_0(t);\ \inf_{i \ge 1}\left(\lambda_i(t) - \mu_i(t)\right)\right\}, \quad t \ge 0. \tag{4.88}$$

Then $\frac{d\mathrm{E}(t;k)}{dt} \ge r(t)$, *and*

$$\mathrm{E}(t;k) \ge k + \int\limits_0^t r(\tau)\, d\tau. \tag{4.89}$$

5 $M_t/M_t/1$ queue

There is a number of papers devoted to this queue, see the references in [29]. Here the number of customers $X(t)$ at time t (the length of the queue) a BDP given by

$$\lambda_n(t) = a(t);\ \mu_n(t) = b(t) \tag{5.90}$$

Bounds for this model were obtained in [39, 40, 42, 43].

Let $\rho = \frac{a_m}{b_m}$ be the traffic intensity. We consider underloaded ($\rho < 1$) and overloaded ($\rho > 1$) cases.

Put

$$\dot{\alpha}(t) = \left(1 - \rho^{1/2}\right)\left(b(t) - \rho^{-1/2} a(t)\right). \qquad (5.91)$$

Put $\delta_i = \rho^{-1/2}$, $d_i = \rho^{-i/2}$, $i \ge 1$. Then we have in Theorem 1 $c = 1 - \rho^{1/2}$, $g = \inf d_k = 1$, $g_i = \sum\limits_{m=0}^{i-1} d_m = \sum\limits_{m=0}^{i-1} \rho^{-m/2}$, $\quad l(t) = \dot{\alpha}(t)$, and Theorem 1 implies the following bound

Proposition 1 . *For underloaded $M_t/M_t/1$ queue ($a_m < b_m$) the following upper bound on the rate of convergence holds:*

$$\left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\| \le 4e^{-\int\limits_s^t \dot{\alpha}(u)du} \sum_{i\ge 1} \left\{\sum_{m=0}^{i-1} \rho^{-m/2}\right\} \left|p_i^1(s) - p_i^2(s)\right|. \qquad (5.92)$$

Remark 4 . *It is easy to see from (5.91) that in the case considered $\dot{\alpha}(t) = \left(\sqrt{a(t)} - \sqrt{b(t)}\right)^2$ which is the "exact" decay function, since the exact value of the spectral gap (decay parameter) for the corresponding homogeneous case is $\dot{\alpha} = \left(\sqrt{a} - \sqrt{b}\right)^2$, see [10].*

Corollary 5 . *Let intensities be periodic with period $T_1 = T_2 = 1$ and $a_m = \int\limits_0^1 a(u)du < b_m = \int\limits_0^1 b(u)du$. Then $\dot{\alpha}(t)$ is also 1-periodic and*

$$\dot{\alpha}_m := \int\limits_0^1 \dot{\alpha}(u)du = \left(\sqrt{a_m} - \sqrt{b_m}\right)^2. \qquad (5.93)$$

Theorem 2 and its Corollary imply the following statements

Proposition 2 . *Let $M_t/M_t/1$ queue be underloaded. Then for any $\varepsilon > 0$ there exists K, such that*

$$\Pr\left(X(t) \le j \mid X(0) = 0\right) \ge \; 1 - K a_m \left\{\sum_{m=0}^{j} \rho^{-m/2}\right\}^{-1} \left(1 + \frac{e^{(\dot{\alpha}-\varepsilon)T_1}}{e^{(\dot{\alpha}-\varepsilon)T_1} - 1}\right). \qquad (5.94)$$

Corollary 6 . *Let $M_t/M_t/1$ queue be underloaded. Then for any $\varepsilon > 0$ there exists K, such that*

$$\mathrm{E}(t;0) \le \frac{K a_m}{\mathrm{W}} \left(1 + \frac{e^{(\dot{\alpha}-\varepsilon)T_1}}{e^{(\dot{\alpha}-\varepsilon)T_1} - 1}\right). \qquad (5.95)$$

where

$$\mathrm{W} = \inf_{i\geq 1}\left\{ i^{-1}\sum_{m=0}^{i-1}\rho^{-m/2}\right\} > 0.$$

Consider the overloaded queue.

The conditions of Theorem 4 hold for $\delta_i = \rho^{-1/2} < 1$, $d_i = \rho^{-i/2}$, $c = \rho^{1/2} - 1$. Then $G = 1$, $\theta(t) = \dot{\alpha}(t)$ and we obtain the following statement

Proposition 3 . *For overloaded $M_t/M_t/1$ queue, ($a_m > b_m$) the following bound of the rate of convergence holds:*

$$\sum_{i=0}^{\infty}\rho^{-i/2}p_i(t) \leq e^{-\int_s^t \dot{\alpha}(u)du}, \tag{5.96}$$

for any $s \geq 0$, $t \geq s$.

Corollary 7 . *If $M_t/M_t/1$ queue is overloaded, then for any $\mathbf{p}(s) \in \Omega$ and any k*

$$p_k(t) \leq \rho^{k/2} e^{-\int_s^t \dot{\alpha}(u)du}. \tag{5.97}$$

The following bound follows from Corollary of Theorem 4.

Proposition 4 . *If $M_t/M_t/1$ queue is overloaded, then*

$$\Pr\left(X(t) \leq j \mid X(0) = k\right) \leq \rho^{(k-j)/2} e^{-\int_0^t \dot{\alpha}(u)du}, \tag{5.98}$$

for any $t \geq 0$.

With the help of (4.89) we get the following bound for the mean of the queue length.

Proposition 5 . *For any k and any $t \geq 0$*

$$\mathrm{E}(t;k) \geq k + \int_0^t \left(a(\tau) - b(\tau)\right) d\tau. \tag{5.99}$$

6 $M_t/M_t/S$ queue

For references see [20], [32], [34], [2]. Here the length of the queue (the number of customers) is BDP $X(t)$ with the intensities

$$\lambda_n(t) = a(t);\ \mu_n(t) = b(t)\min(n,S). \tag{6.100}$$

Bounds for this process were obtained in [39, 40], [42], [43]. The estimates of probabilities for the perturbed $M_t/M_t/S$ queue were studied in [45].

Let $\rho = \frac{a_m}{Sb_m}$ be the traffic intensity. We consider underloaded ($\rho < 1$) and overloaded ($\rho > 1$) cases.

Proposition 6 . *Let $M_t/M_t/S$ queue be underloaded. Then (3.43) holds for*

$$c = \min\left(\frac{1}{S}, \frac{\Delta - 1}{\Delta}\right), \quad \Delta > 1. \tag{6.101}$$

Proposition 7 . *Let $M_t/M_t/S$ queue be overloaded. Then (4.73) holds for*

$$c = \frac{\Delta - 1}{\Delta}, \quad \Delta > 1. \tag{6.102}$$

First, let the queue be underloaded. Put $\Delta = \rho^{-1/2}$. Then $c = \min\left(\frac{1}{S}, 1 - \rho^{1/2}\right)$. We consider the cases of heavy traffic $\rho \geq \left(\frac{S-1}{S}\right)^2$, $\quad c = 1 - \rho^{1/2}$, and light traffic $0 \leq \rho < \left(\frac{S-1}{S}\right)^2$, $\quad c = \frac{1}{S}$.

Put

$$\dot\alpha(t) = c\left(Sb(t) - \rho^{-1/2}a(t)\right). \tag{6.103}$$

Put $\delta_i = \rho^{-1/2}$, $d_i = \rho^{-i/2}$, $i \geq 1$. Then we have in Theorem 1, $g = \inf d_k = 1$, $q_i = \sum\limits_{m=0}^{i-1} d_m = \sum\limits_{m=0}^{i-1} \rho^{-m/2}$, $\quad l(t) = \dot\alpha(t)$, and by Theorem 1 we have

Proposition 8 . *For underloaded $M_t/M_t/S$ queue ($a_m < Sb_m$), the following bound for the rate of convergence is valid:*

$$\left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\| \leq 4e^{-\int_s^t \dot\alpha(u)du} \sum_{i\geq 1}\left\{\sum_{m=0}^{i-1} \rho^{-m/2}\right\} \left|p_i^1(s) - p_i^2(s)\right|. \tag{6.104}$$

Remark 5 . *For underloaded $M_t/M_t/S$ queue with heavy traffic the exact "decay function" is $\dot\alpha(t) = \left(\sqrt{a(t)} - \sqrt{Sb(t)}\right)^2$ because the exact value of spectral gap (decay parameter) for homogeneous case with heavy traffic is $\dot\alpha = \left(\sqrt{a} - \sqrt{Sb}\right)^2$, see [10]. Moreover, for $S = 2$ the bound between light and heavy traffics is also exact.*

Corollary 8 . *Let intensities be periodic with period* $T_1 = T_2 = 1$ *and* $a_m = \int\limits_0^1 a(u)\,du < Sb_m = S\int\limits_0^1 b(u)\,du$. *Then* $\dot\alpha(t)$ *is also* 1*-periodic and*
in the case of heavy traffic

$$\dot\alpha_m := \int\limits_0^1 \dot\alpha(u)du = \left(\sqrt{a_m} - \sqrt{Sb_m}\right)^2. \tag{6.105}$$

in the case of light traffic

$$\dot\alpha_m := \int\limits_0^1 \dot\alpha(u)du = b_m - \sqrt{Sa_m b_m}. \tag{6.106}$$

Theorem 2 and its Corollary imply the following two statements

Proposition 9 . *Let* $M_t/M_t/S$ *queue be underloaded. Then for any* $\varepsilon > 0$ *there exists* K*, such that*

$$\Pr\left(X(t) \le j \mid X(0) = 0\right) \ge 1 - Ka_m \left\{\sum_{m=0}^{j} \rho^{-m/2}\right\}^{-1} \left(1 + \frac{e^{(\dot\alpha-\varepsilon)T_1}}{e^{(\dot\alpha-\varepsilon)T_1} - 1}\right). \tag{6.107}$$

Corollary 9 . *Let* $M_t/M_t/S$ *queue be underloaded. Then for any* $\varepsilon > 0$ *there exists* K*, such that*

$$\mathrm{E}(t;0) \le \frac{Ka_m}{\mathrm{W}} \left(1 + \frac{e^{(\dot\alpha-\varepsilon)T_1}}{e^{(\dot\alpha-\varepsilon)T_1} - 1}\right). \tag{6.108}$$

where

$$\mathrm{W} = \inf_{i\ge1}\left\{i^{-1}\sum_{m=0}^{i-1}\rho^{-m/2}\right\} > 0.$$

Let now $M_t/M_t/S$ queue be overloaded. Put $\Delta = \rho^{1/2}$. Then $c = \rho^{1/2} - 1$. The conditions of Theorem 4 hold for $\delta_i = \rho^{-1/2} < 1$, $d_i = \rho^{-i/2}$, $c = 1 - \rho^{1/2}$. Then $G = 1$, $\theta(t) = \dot\alpha(t)$ and we obtain

Proposition 10 . *For overloaded* $M_t/M_t/S$ *queue* ($a_m > Sb_m$) *the following bound for the rate of convergence is valid:*

$$\sum_{i=0}^{\infty} \rho^{-i/2} p_i(t) \le e^{-\int\limits_s^t \dot\alpha(u)du}, \tag{6.109}$$

for any $s \ge 0$, $t \ge s$.

Corollary 10 . *If $M_t/M_t/S$ queue is overloaded, then for any $\mathbf{p}(s) \in \Omega$ and any k*

$$p_k(t) \le \rho^{k/2} e^{-\int_s^t \dot{\alpha}(u)du}. \tag{6.110}$$

The following bound follows from Corollary of Theorem 4.

Proposition 11 . *If $M_t/M_t/S$ queue is overloaded, then*

$$\Pr(X(t) \le j \mid X(0) = k) \le \rho^{(k-j)/2} e^{-\int_0^t \dot{\alpha}(u)du}, \tag{6.111}$$

for any $t \ge 0$.

¿From (4.89) one gets the following simple bound for the mean of the length of the queue

Proposition 12 . *For any k and any $t \ge 0$*

$$\mathrm{E}(t;k) \ge k + \int_0^t (a(\tau) - Sb(\tau))\, d\tau. \tag{6.112}$$

7 A multiserver queue with discouragements

This queueing model was studied in [9, 30, 31, 33]. We have here BDP with the intensities $\lambda_n(t) = \lambda_n a(t)$; $\mu_n(t) = \mu_n b(t)$, where

$$\lambda_n = \begin{cases} \lambda & \text{if } n < S \\ \frac{\lambda}{n-S+2} & \text{if } n \ge S \end{cases}, \ \mu_n = \begin{cases} n\mu & \text{if } n \le S \\ S\mu & \text{if } n > S \end{cases}. \tag{7.113}$$

So, in the case considered $\lim\limits_{n\to\infty} \lambda_n = 0$, which says that a straightforward application of Theorem 1 is impossible. Put

$$\delta_k = \begin{cases} 1 & \text{if } k < S \\ 1+\varepsilon & \text{if } k \ge S \end{cases} \tag{7.114}$$

where $\varepsilon \in (0;1)$. Then we have

$$\alpha_k(t) = b(t), \quad k < S-1;$$

$$\alpha_{S-1}(t) = b(t) - a(t)\left(\frac{1+\varepsilon}{2} - 1\right),$$

$$\alpha_{n+S}(t) = Sb(t)\left(1 - \frac{1}{1+\varepsilon}\right) - a(t)\left(\frac{1+\varepsilon}{n+3} - \frac{1}{n+2}\right), \quad n = k - S \geq 0. \tag{7.115}$$

Put

$$c_n = \frac{(n+2)\varepsilon - 1}{(n+2)(n+3)} \tag{7.116}$$

If $(n+2) \leq \varepsilon^{-1}$, then $c_n \leq 0$; if $(n+2) > \varepsilon^{-1}$, then $c_n < \frac{\varepsilon}{n+3} < \frac{\varepsilon^2}{1+\varepsilon} < \frac{S\varepsilon^2}{1+\varepsilon}$. Finally we obtain for any k the following bound

$$\alpha_k(t) \geq \dot{\alpha}(t) = \frac{S\varepsilon}{1+\varepsilon}\left(b(t) - \varepsilon a(t)\right). \tag{7.117}$$

Let $b_m > 0$ and let $\varepsilon \in (0;1)$, be such that $b_m - \varepsilon a_m > 0$. Then applying the estimates of Theorem 1 we obtain

Proposition 13 .

$$\left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\| \leq 4e^{-\int_s^t \dot{\alpha}(u)du} \sum_{i\geq 1} g_i \left|p_i^1(s) - p_i^2(s)\right|, \tag{7.118}$$

where $q_i = \sum_{m=0}^{i-1} \prod_{k=0}^{m} \delta_k$.

Also, Corollary 2 and Theorem 2 imply, under the same assumption as above,

Proposition 14 *Let* $b_m > 0$. *Then there exists* K, *such that*

$$\Pr\left(X(t) \leq j \mid X(0) = 0\right) \geq 1 - q_{j+1}^{-1} K\lambda a_m \left(1 + \frac{e^{(\dot{\alpha}-\varepsilon)T_1}}{e^{(\dot{\alpha}-\varepsilon)T_1} - 1}\right). \tag{7.119}$$

Corollary 11 . *Let* $b_m > 0$. *Then there exists* K, *such that*

$$\mathrm{E}(t;0) \leq \frac{Ka_m}{\mathrm{W}}\left(1 + \frac{e^{(\dot{\alpha}-\varepsilon)T_1}}{e^{(\dot{\alpha}-\varepsilon)T_1} - 1}\right). \tag{7.120}$$

where $\mathrm{W} = \inf_{i\geq 1}(q_i/i) > 0$.

8 $M_t/M_t/S/S$ queue

There is a large number of investigations of the $M/M/S/S$ queue, see references in [24]. In particular, this queue is a subject of some new papers, see [13], [36], [35]. In general (nonstationary case) the length of a queue $X(t)$ is a BDP on the state space $E = \{0, 1, \dots, S\}$ with the intensities

$$\lambda_n(t) = a(t); \ \mu_n(t) = nb(t) \tag{8.121}$$

The first bounds for the general case were obtained in [38], see also [42, 43]; some of results of [38] were repeated in [28], and in the last paper non-Markovian case was also studied.

Consider the condition

$$a_m + b_m > 0. \tag{8.122}$$

One can see that BDP is not exponentially ergodic if (8.122) is not fulfilled.

Let (8.122) holds.

The first case. Let

$$b_m > 0. \tag{8.123}$$

Put $\delta_k = 1, \ k = 1, \dots, S-1$. Then $d_k = 1, \ g = \min d_k = 1, \ G = \max d_k = 1$, and

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \delta_{k+1}\lambda_{k+1}(t) - \delta_k^{-1}\mu_k(t) = \begin{cases} b(t) & if k < S-1 \\ b(t) + a(t) & if k = S-1 \end{cases} \tag{8.124}$$

and

$$\tilde{\alpha}(t) = \inf_k \alpha_k(t) = b(t), \tag{8.125}$$

$$\tilde{\beta}(t) = \sup_k \alpha_k(t) = a(t) + b(t), \tag{8.126}$$

$$\tilde{\zeta}(t) = \sup_k \left\{\lambda_k(t) + \mu_{k+1}(t) + \delta_{k+1}\lambda_{k+1}(t) + \delta_k^{-1}\mu_k(t)\right\} \le 2a(t) + (2S-1)b(t). \tag{8.127}$$

Now Theorem 3 implies

Proposition 15 . *Let the condition (8.123) be satisfied. Then for any* $s \geq 0$, $t \geq s$, $\mathbf{p}^1(s) \in \Omega$, $\mathbf{p}^2(s) \in \Omega$

$$e^{-\int_s^t \tilde{\zeta}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_{1D} \leq \left\|\mathbf{p}^{(1)}(t) - \mathbf{p}^{(2)}(t)\right\|_{1D} \leq e^{-\int_s^t \tilde{\alpha}(u)du} \left\|\mathbf{p}^{(1)}(s) - \mathbf{p}^{(2)}(s)\right\|_{1D}, \tag{8.128}$$

and

$$\frac{1}{4S} e^{-\int_s^t \tilde{\zeta}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_1 \leq \left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\|_1 \leq 4S e^{-\int_s^t \tilde{\alpha}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_1. \tag{8.129}$$

Remark 6 . *If the intensities are constant, then the bounds (8.128) and (8.129) immediately imply the inequality (17) in the paper [13]. It is interesting to note that such bound in general case was found about ten years ago in [38]. The cutoff property, which is studied in [13] and [36] can be naturally investigated by our methods. It will be a subject of a separate paper.*

Proposition 16 . *Let the condition (8.123) be satisfied. Then for any* $s \geq 0$, $t \geq s$, *and* $\mathbf{p}^1(s) \leq \mathbf{p}^2(s)$ *the following bounds hold*

$$\left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\|_{1D} \geq e^{-\int_s^t \tilde{\beta}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_{1D}, \tag{8.130}$$

and

$$\left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\|_1 \geq \frac{1}{4S} e^{-\int_s^t \tilde{\beta}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_1. \tag{8.131}$$

Consider now the bounds for the length of queue. We have here $W = \inf_{i\geq 1} \frac{g_i}{i} = \inf_{i \geq 1} \frac{1}{i} \left\{ \sum_{m=0}^{i-1} \prod_{k=0}^{m} \delta_k \right\} = 1$.

Theorem 2 and its Corollary imply the following statetment.

Proposition 17 *Let the condition (8.123) be satisfied. Then for any* $\varepsilon > 0$ *there exists* K, *such that*

$$\Pr\left(X(t) \leq j \mid X(0) = 0\right) \geq 1 - \frac{K a_m}{j+1} \left(1 + \frac{e^{(b_m - \varepsilon)T_1}}{e^{(b_m-\varepsilon)T_1} - 1}\right). \tag{8.132}$$

and

$$\mathrm{E}(t;0) \le K a_m \left(1 + \frac{e^{(b_m-\varepsilon)T_1}}{e^{(b_m-\varepsilon)T_1} - 1}\right). \tag{8.133}$$

On the other hand, the estimate of the mean of the number of customers in the queue can be obtained in the following way:

$$\frac{d\mathrm{E}(t;k)}{dt} = \lambda_0(t)\,p_0 + \sum_{i=1}^{S}(\lambda_i(t) - \mu_i(t))\,p_i = a(t)\sum_{i=0}^{S-1} p_i - b(t)\sum_{i=0}^{S} ip_i \le$$

$$a(t)\sum_{i=0}^{S} p_i - b(t)\,\mathrm{E}(t;k) = a(t) - b(t)\,\mathrm{E}(t;k). \tag{8.134}$$

By integration of this inequality we get

Proposition 18 *The following bound for the mean of the length of $M_t/M_t/S/S$ queue holds:*

$$\mathrm{E}(t;k) \le k e^{-\int_0^t b(u)du} + \int_0^t a(\tau)\, e^{-\int_\tau^t b(u)du} d\tau. \tag{8.135}$$

Remark 7 . *The bound (8.135) seems to be sufficiently sharp for the case of light traffic. It is interesting to study its sharpness in the general case.*

The second case. Let now

$$a_m > 0. \tag{8.136}$$

Put $\delta_k = \delta = \frac{S-1}{S} < 1,\ k = 1, \dots, S-1$. Then

$$\begin{aligned} \alpha_k(t) &= \lambda_k(t) + \mu_{k+1}(t) - \delta_{k+1}\lambda_{k+1}(t) - \delta_k^{-1}\mu_k(t) \\ &= \begin{cases} (1-\delta)\,a(t) + \left(k - \frac{k-1}{\delta}\right) b(t) & ,if\ k < S-1 \\ a(t) & ,if\ k = S-1 \end{cases} \end{aligned} \tag{8.137}$$

and

$$\breve{\alpha}(t) = \inf_k \alpha_k(t) \ge (1-\delta)\,a(t) = \frac{a(t)}{S}, \tag{8.138}$$

$$\breve{\beta}(t) = \sup_k \alpha_k(t) \le a(t) + b(t) = \tilde{\beta}(t), \tag{8.139}$$

$$\breve{\zeta}(t) = \sup_k \left\{\lambda_k(t) + \mu_{k+1}(t) + \delta_{k+1}\lambda_{k+1}(t) + \delta_k^{-1}\mu_k(t)\right\} \le (1+\delta)\,a(t) + 2Sb(t). \tag{8.140}$$

Then by virtue of Theorem 3, we have

Proposition 19 *Let the condition (8.136) hold. Then for any $s \geq 0$, $t \geq s$, and any* $\mathbf{p}^1(s) \in \Omega$, $\mathbf{p}^2(s) \in \Omega$

$$e^{-\int_s^t \check{\zeta}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_{1D} \leq \left\|\mathbf{p}^{(1)}(t) - \mathbf{p}^{(2)}(t)\right\|_{1D} \leq e^{-\int_s^t \check{\alpha}(u)du} \left\|\mathbf{p}^{(1)}(s) - \mathbf{p}^{(2)}(s)\right\|_{1D}, \tag{8.141}$$

and

$$\frac{S-1}{4S^2} e^{-\int_s^t \check{\zeta}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_1 \leq \left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\|_1 \tag{8.142}$$

$$\leq \frac{4S^2}{S-1} e^{-\int_s^t \check{\alpha}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_1.$$

Proposition 20 *Let the condition (8.136) hold. Then for any $s \geq 0$, $t \geq s$, and* $\mathbf{p}^1(s) \leq \mathbf{p}^2(s)$ *the following bounds are valid*

$$\left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\|_{1D} \geq e^{-\int_s^t \check{\beta}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_{1D}, \tag{8.143}$$

and

$$\left\|\mathbf{p}^1(t) - \mathbf{p}^2(t)\right\|_1 \geq \frac{S-1}{4S^2} e^{-\int_s^t \check{\beta}(u)du} \left\|\mathbf{p}^1(s) - \mathbf{p}^2(s)\right\|_1. \tag{8.144}$$

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