

Mean Characteristics of Markov Queueing Systems¹

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Abstract—Consideration is given to queueing systems described by nonstationary birth-death processes with rates close to periodic. Questions connected with the existence and design of limiting mean characteristics are studied. Some examples of designing the means for concrete queueing systems are considered.

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1. INTRODUCTION

Markov models of the queueing theory described by birth-death processes (BDP) have been investigated and applied for a long time (see, e.g., [1, 2]). At the same time, creation of new methods to study BDP and enlargement of the range of the solved problems invoked appearance of a new series of works [3–11].

More realistic models of queueing systems described by nonstationary Markov chains are actively studied in B.V. Gnedenko's works, early [12, 13] and the following ones [14–18]. As a rule, it is impossible to design the limiting mode and find explicit formulas for state probabilities of these models; therefore, the main interest is focused on questions of characteristic approximation for these systems [19–22].

The interest in nonstationary Markov models of queueing systems has increased in the last years after the creation of new investigation methods (e.g., [23–27]).

Two main characteristics, limiting mean and double mean, are introduced and studied for the processes with periodic rates in [27].

In this paper, we study a classic queueing system (QS) $M_n(t)/M_n(t)/S$ with rates of customer arrival and service close to periodic, obtain estimates of the means, indicate the technique for their approximate computation, and consider several examples. The work is an expanded and supplemented variant of the preliminary report [28].

2. BASIC NOTION

Let $X(t)$, $t \geq 0$, is an “auxiliary” nonhomogeneous BDP with the state space $E = \{0, 1, \dots\}$, whose rates of birth $\lambda_n(t)$, $t \geq 0$, and death $\mu_n(t)$, $t \geq 0$, $n \in E$, are periodic functions of time t with the period equal to unity. Let us assume that the following conditions (standard for the situation with variable rates) are fulfilled for the auxiliary process:

$$\lambda_n(t) = \nu_n a(t), \quad \mu_n(t) = \eta_n b(t), \quad t \geq 0, \quad n \in E, \quad (1)$$

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where

$$\eta_0 = 0, \quad 0 \leq \nu_n \leq L < \infty, \quad 0 < \eta_{n+1} \leq M < \infty, \quad n \in E. \quad (2)$$

Let $a(t)$ and $b(t)$ are nonnegative, 1-periodic, and integrable by $[0, 1]$. Moreover, to obtain simpler estimates, we assume that for all $t \in [0, 1]$

$$a(t) \leq \mathbf{a} \quad \text{and} \quad b(t) \leq \mathbf{b}. \quad (3)$$

Let us now consider the main (“perturbed”) BDP $X^*(t)$, $t \geq 0$, with the same state space E , birth rates $\lambda_n^*(t)$, $t \geq 0$, and death rates $\mu_n^*(t)$, $t \geq 0$, $n \in E$, for which are fulfilled similar standard conditions; instead of 1-periodicity we assume that

$$\lambda_n^*(t) - \lambda_n(t) = \widehat{\lambda}_n(t), \quad \mu_n^*(t) - \mu_n(t) = \widehat{\mu}_n(t), \quad (4)$$

where “perturbations” of the rates are small: $|\widehat{\lambda}_n(t)| \leq \varepsilon$, $|\widehat{\mu}_n(t)| \leq \varepsilon$ with all $t \geq 0$ and $n \in E$.

Let us use the notion of the limiting mean and double mean in [27]. Namely, we shall consider that the BDP $X(t)$ has a limiting mean $\phi(t)$, if

$$\lim_{t \rightarrow \infty} |E \{X(t) | X(0) = k\} - \phi(t)| = 0$$

for any $k \in E$. Here $E \{X(t) | X(0) = k\}$ is a mean for the process at the instant t on condition that $X(0) = k$.

The double mean for $X(t)$ is a limit

$$E = \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t E \{X(u) | X(0) = k\} du,$$

if it exists and does not depend on k .

3. DESIGN AND ESTIMATION OF LIMITING MEANS

As in the previous authors’ works, we assume that state probability vectors of the considered process are described by the corresponding direct Kolmogorov systems (see (A.1) and (A.2)).

The method devised earlier by us (see, e.g., [17, 18]) consists of several steps: instead of the initial system of differential equations, we consider the “reduced” one (with the excluded zero state); then the matrix of the new system is reduced to the most convenient form by some special renormalization; finally, the rate of convergence is estimated by the logarithmic norm introduced in [29] and studied in [30].

To fulfill necessary renormalizations, the key step is the design of an auxiliary “renormalizing” sequence that is an analog to the Lyapunov function and does not have the probability sense. The questions related to the design of these sequences and the study of their properties in different situations are considered in detail in [31].

Namely, let there is some sequence of positive numbers $1 = d_{-1} = d_0 \leq d_1 \leq \dots$. Let

$$\alpha_k(t) = \lambda_k(t) + \mu_{k+1}(t) - \frac{d_{k+1}}{d_k} \lambda_{k+1}(t) - \frac{d_{k-1}}{d_k} \mu_k(t), \quad k \geq 0, \quad (5)$$

and introduce some auxiliary quantities:

$$\Gamma = \sup_k \left(2 + \frac{d_{k+1}}{d_k} + \frac{d_{k-1}}{d_k} \right), \quad \alpha(t) = \inf_k \alpha_k(t), \quad \alpha^* = \int_0^1 \alpha(t) dt,$$

$$M_0 = \sup_{|t-s| \leq 1} \int_s^t \alpha(u) du, \quad M = e^{M_0 + \alpha^*}, \quad W = \inf_k \frac{\sum_{i=0}^{k-1} d_i}{k}. \tag{6}$$

Theorem 1. *Let there exist a sequence $\{d_i\}$ such that $\alpha^* > 0$, $\inf_{k \geq 1} \frac{d_{k-1}}{k} > 0$, and $\Gamma < \infty$ and $\alpha^* - \Gamma\epsilon > 0$. Then the auxiliary BDP $X(t)$ has the 1-periodic limiting mean $\phi(t)$; for any starting condition $X^*(0) = k$ and all $t \geq 0$ is true the estimate*

$$|E \{X^*(t) | X^*(0) = k\} - \phi(t)| \leq \frac{M}{W} \left(e^{(-\alpha^* + \Gamma\epsilon)t} \left(\frac{M\alpha\nu_0}{\alpha^*} + \sum_{i=0}^k d_i \right) + \frac{\epsilon}{\alpha^* - \Gamma\epsilon} \left(1 + \frac{\Gamma M\alpha\nu_0}{\alpha^*} \right) \right). \tag{7}$$

Theorem 2. *Let the conditions of Theorem 1 are fulfilled. Then the BDP $X^*(t)$ has the limiting mean $\phi^*(t)$; is fulfilled the inequality*

$$\overline{\lim}_{t \rightarrow \infty} |\phi(t) - \phi^*(t)| \leq \frac{M\epsilon}{W(\alpha^* - \Gamma\epsilon)} \left(1 + \frac{\Gamma M\alpha\nu_0}{\alpha^*} \right). \tag{8}$$

Remark 1. It is to be noted that unlike the case when rates of BDP are 1-periodic, the limiting mean $\phi^*(t)$ should not necessarily be periodic for the BDP $X^*(t)$ and there can be no double mean at all. However, some weaker analog of the double mean can be estimated for this situation as well.

Let us introduce notations (upper and lower) for the BDP $X^*(t)$:

$$\overline{E}^* = \overline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t E \{X^*(u) | X^*(0) = k\} du,$$

$$\underline{E}^* = \underline{\lim}_{t \rightarrow \infty} \frac{1}{t} \int_0^t E \{X^*(u) | X^*(0) = k\} du.$$

It is to be noted that the double mean E is obtained for BDP with 1-periodic rates as a period mean of the limiting mean: $E = \int_0^1 \phi(t) dt$. Then from Theorem 2 we obtain Corollary 1.

Corollary 1. *Let are fulfilled the conditions of Theorem 1. Then the BDP $X^*(t)$ has the upper and lower means, and is fulfilled the inequality*

$$E - \frac{M\epsilon}{W(\alpha^* - \Gamma\epsilon)} \left(1 + \frac{\Gamma M\alpha\nu_0}{\alpha^*} \right) \leq \underline{E}^* \leq \overline{E}^* \leq E + \frac{M\epsilon}{W(\alpha^* - \Gamma\epsilon)} \left(1 + \frac{\Gamma M\alpha\nu_0}{\alpha^*} \right). \tag{9}$$

Let us now study questions related to the design of the limiting, upper, and lower means for the BDP $X^*(t)$. Let us introduce some auxiliary notion. Let $X_N(t)$, $t > 0$ is a “trimmed” BDP with the state space $E_N = \{0, 1, 2, \dots, N\}$ and 1-periodic rates of birth $\lambda_n(t)$, $n \in E_{N-1}$, and

death $\mu_n(t)$, $n \in E_N$ (and the rate matrix $A_N(t)$). Let us denote corresponding characteristics of this process by the index N .

Let

$$W_N = \inf_{k \geq 0} \frac{\sum_{i=0}^k d_{N-1+i}}{N+k}. \tag{10}$$

Let us consider the approximate computation $E \{X^*(t) | X^*(0) = k\}$, that will enable computing the limiting mean $\phi^*(t)$.

Theorem 3. *Let the conditions of Theorem 1 are fulfilled. Then with any N, k and all $t \geq 0$ is fulfilled the inequality*

$$\begin{aligned} & |E \{X^*(t) | X^*(0) = k\} - E \{X_N(t) | X(0) = 0\}| \\ & \leq \frac{M^2 a \nu_0}{W \alpha^*} e^{-\alpha^* t} + \frac{3t M a \nu_0 (La + Mb)}{W_N \alpha^*} \\ & + \frac{M}{W} \left(e^{(-\alpha^* + \Gamma \epsilon)t} \left(\frac{M a \nu_0}{\alpha^*} + \sum_{i=0}^k d_i \right) + \frac{\epsilon}{\alpha^* - \Gamma \epsilon} \left(1 + \frac{\Gamma M a \nu_0}{\alpha^*} \right) \right). \end{aligned} \tag{11}$$

The following statement makes it possible to compute the upper and lower means approximately for $X^*(t)$.

Theorem 4. *Let the conditions of Theorem 1 are fulfilled. Then with any N and all $t \geq 0$ is fulfilled the inequality*

$$\begin{aligned} & \left| \underline{E}^* - \int_t^{t+1} E \{X_N(u) | X(0) = 0\} du \right| \\ & \leq \frac{M \epsilon}{W (\alpha^* - \Gamma \epsilon)} \left(1 + \frac{\Gamma M a \nu_0}{\alpha^*} \right) + \frac{M^2 a \nu_0}{W \alpha^*} e^{-\alpha^* t} + \frac{3(t+1) M a \nu_0 (La + Mb)}{W_N \alpha^*}. \end{aligned} \tag{12}$$

The same estimate is also true for \underline{E}^* .

Remark 2. Similar results can be easily obtained for queueing systems described by

- (1) birth-death processes whose rates are close to periodic with the arbitrary period $T \neq 1$;
- (2) processes with the finite number of states (in particular, these are models of the type $M(t)/M(t)/K/L$).

4. EXAMPLES

Let us consider several examples of computing the means for concrete QS.

Example 1. The number of customers in the queueing system $M(t)/M(t)/1$.

Let $a^*(t) = 1 + 0.5 \sin 2\pi t + 10^{-9} \sin^2 t$, and $b^*(t) = 6 + \cos 4\pi t + 10^{-9} \cos \sqrt{t}$. Assuming $a(t) = 1 + 0.5 \sin 2\pi t$, $b(t) = 6 + \cos 4\pi t$, we have $\epsilon = 10^{-9}$ and, moreover, $\nu_n = \eta_n = 1$ ($n \geq 1$). Then $L = M = 1$, $a = 1.5$, $b = 7$. Assuming $d_k = 2^k$, $k \geq 0$, we obtain $\alpha(t) = 2 - 0.5 \sin 2\pi t + 0.5 \cos 2\pi t$,

$$\alpha^* = \int_0^1 \alpha(t) dt = 2, M_0 = \sup_{|t-s| \leq 1} \int_s^t \alpha(u) du \leq 3, M = e^{M_0 + \alpha^*} = e^5, W = \inf_k \frac{\sum_{i=0}^{k-1} d_i}{k} = 1$$
 and

$$W_N = \inf_{k \geq 0} \frac{\sum_{i=0}^k 2^{N-1+i}}{N+k} = \frac{2^{N-1}}{N}.$$

Under $t = 11, N = 46$, we obtain Fig. 1.

In Fig. 1 is represented $\phi(t), t \in [11, 12]$, accurate within 10^{-5} . Hence, this figure gives $\phi^*(t)$ under greater t accurate within 10^{-4} .

$\bar{E}^* \approx 0.2023$ accurate within 10^{-4} .

Example 2. The number of customers in the queueing system $M(t)/M(t)/2$.

Let $a^*(t) = 1 + \sin 2\pi t + 10^{-10} \sin t^2$, and $b^*(t) = 4 + 2 \cos 2\pi t + 10^{-10} \cos \sqrt{t}$. Assuming $a(t) = 1 + \sin 2\pi t, b(t) = 4 + 2 \cos 2\pi t$, we have $\epsilon = 10^{-10}$, and $\nu_n = \eta_1 = 1 (n \geq 1)$ and $\eta_n = 2 (n \geq 2)$. Then $L = 1, M = 2, a = 2$ and $b = 6$. Assuming $d_k = 2^k, k \geq 0$, we obtain $\alpha(t) = 3 - \sin 2\pi t + 2 \cos 2\pi t, \alpha^* = \int_0^1 \alpha(t) dt = 3,$

$$M_0 = \sup_{|t-s| \leq 1} \int_s^t \alpha(u) du = 3 + \frac{1}{2\pi}, M = e^{M_0 + \alpha^*} = e^{6 + \frac{1}{2\pi}} < 500,$$

$$W = \inf_k \frac{\sum_{i=0}^{k-1} d_i}{k} = 1 \text{ and } W_N = \inf_{k \geq 0} \frac{\sum_{i=0}^k 2^{N-1+i}}{N+k} = \frac{2^{N-1}}{N}.$$

Under $t = 9, N = 41$, we obtain Fig. 2.

In Fig. 2 is represented $\phi(t), t \in [9, 10]$, accurate within 10^{-5} . Hence, this figure gives $\phi^*(t)$ under greater t accurate within 10^{-4} .

$\bar{E}^* \approx 0.2945$ accurate within 10^{-4} .

Example 3. The number of customers in the queueing system $M(t)/M(t)/100$.

Let $a^*(t) = 1 + \sin 2\pi t + 10^{-11} \sin t^2$, and $b^*(t) = 4 + 2 \cos 2\pi t + 10^{-11} \cos \sqrt{t}$. Assuming $a(t) = 1 + \sin 2\pi t, b(t) = 4 + 2 \cos 2\pi t$, we have $\epsilon = 10^{-11}$, and $\nu_n = 1, \eta_n \leq 100, L = 1, M = 100, a = 2$ and $b = 6$. Let $d_k = 1.5^k, k \geq 0$, then we obtain $\alpha_k(t) \geq 3.5 - 0.5 \sin 2\pi t + 2 \cos 2\pi t \geq 1$ under all k , then $\alpha(t) \geq 1, \alpha^* = \int_0^1 \alpha(t) dt = 1, M_0 = \sup_{|t-s| \leq 1} \int_s^t \alpha(u) du \leq 5 M = e^{M_0 + \alpha^*} = e^6 < 500.$

$$\text{Then we have } W = \inf_k \frac{\sum_{i=0}^{k-1} d_i}{k} = 1 \text{ and } W_N = \inf_{k \geq 0} \frac{\sum_{i=0}^k 1.5^{N-1+i}}{N+k} = \frac{1.5^{N-1}}{N}.$$

Under $t = 25, N = 85$, we obtain Fig. 3.

In Fig. 3 is represented $\phi(t), t \in [25, 26]$, accurate within 10^{-5} . Hence, this figure gives $\phi^*(t)$ under greater t accurate within 10^{-4} .

$\bar{E}^* \approx 0.2885$ accurate within 10^{-4} .

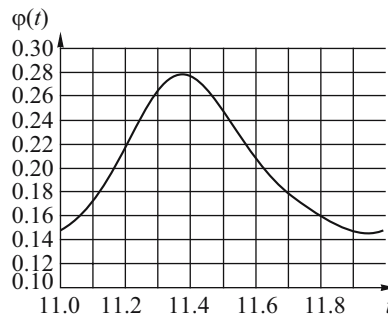


Fig. 1.

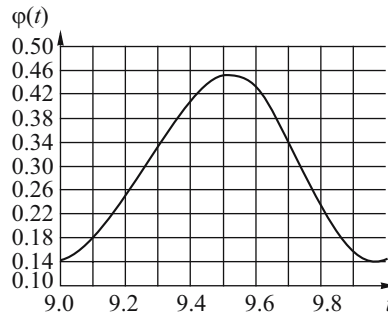


Fig. 2.

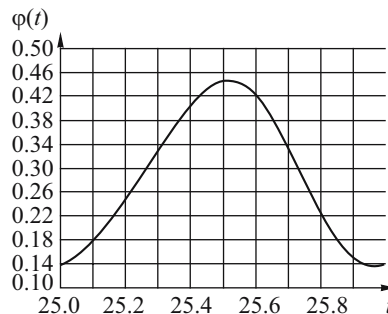


Fig. 3.

APPENDIX

Proof of Theorem 1. Let us consider a direct Kolmogorov system of equations for each of the processes under study:

$$\frac{d\mathbf{p}}{dt} = A(t) \mathbf{p}, \quad \mathbf{p} = \mathbf{p}(t) = (p_0(t), p_1(t), \dots)^T, \quad t \geq 0, \tag{A.1}$$

and

$$\frac{d\mathbf{p}^*}{dt} = A^*(t)\mathbf{p}^*, \quad \mathbf{p}^* = \mathbf{p}^*(t), \quad t \geq 0. \tag{A.2}$$

Let us change $p_0 = 1 - \sum_{i=1}^{\infty} p_i$ in (A.1) and $p_0^* = 1 - \sum_{i=1}^{\infty} p_i^*$ in (A.2). Then we obtain respectively:

$$\frac{d\mathbf{z}}{dt} = B(t)\mathbf{z} + \mathbf{f}(t), \tag{A.3}$$

where

$$B(t) = \begin{pmatrix} -(\lambda_0(t) + \lambda_1(t) + \mu_1(t)) & (\mu_2(t) - \lambda_0(t)) & -\lambda_0(t) & -\lambda_0(t) & \cdots & \cdots \\ \lambda_1(t) & -(\lambda_2(t) + \mu_2(t)) & \mu_3(t) & 0 & 0 & \cdots \\ 0 & \lambda_2(t) & -(\lambda_3(t) + \mu_3(t)) & \mu_4 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$\mathbf{f}(t) = \begin{pmatrix} \lambda_0(t) \\ 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix} \tag{A.4}$$

and

$$\frac{d\mathbf{z}^*}{dt} = B^*(t)\mathbf{z}^* + \mathbf{f}^*(t), \tag{A.5}$$

where $B^*(t), \mathbf{f}^*(t)$ are defined in the similar manner.

Let us rewrite (A.3) as follows:

$$\frac{d\mathbf{z}}{dt} = B^*(t)\mathbf{z} + (B(t) - B^*(t))\mathbf{z} + \mathbf{f}(t). \tag{A.6}$$

Then using the known formulas (see, e.g., [30]), we obtain:

$$\mathbf{z}^*(t) = U(t,0)\mathbf{z}^*(0) + \int_0^t U(t,\tau)\mathbf{f}^*(\tau)d\tau \tag{A.7}$$

from the Eq. (A.5) and

$$\mathbf{z}(t) = U(t,0)\mathbf{z}(0) + \int_0^t U(t,\tau)(B(\tau) - B^*(\tau))\mathbf{z}(\tau)d\tau + \int_0^t U(t,\tau)\mathbf{f}(\tau)d\tau \tag{A.8}$$

from the Eq. (A.6). Here $U(t,\tau)$ is a Cauchy operator for the Eq. (A.5).

Then in any of the norms studied further, we obtain the inequality:

$$\begin{aligned} \|\mathbf{z}(t) - \mathbf{z}^*(t)\| &\leq \|U(t,0)\| \|\mathbf{z}(0) - \mathbf{z}^*(0)\| \\ &+ \int_0^t \|U(t,\tau)\| \|B(\tau) - B^*(\tau)\| \|\mathbf{z}(\tau)\| d\tau + \int_0^t \|U(t,\tau)\| \|\mathbf{f}(\tau) - \mathbf{f}^*(\tau)\| d\tau. \end{aligned} \tag{A.9}$$

Let us consider a matrix of the form

$$D = \begin{pmatrix} d_0 & d_0 & \cdots & \cdots & d_0 & \cdots \\ 0 & d_1 & \cdots & \cdots & d_1 & \cdots \\ 0 & 0 & d_2 & \cdots & d_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & d_{N-1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}, \tag{A.10}$$

where d_i are some positive numbers, and the corresponding norm

$$\|\mathbf{x}\|_{1D} = \|D\mathbf{x}\|_1. \tag{A.11}$$

Then

$$D^{-1} = \begin{pmatrix} d_0^{-1} & -d_1^{-1} & 0 & \cdots & 0 & \cdots \\ 0 & d_1^{-1} & -d_2^{-1} & 0 & \cdots & \cdots \\ 0 & 0 & d_2^{-1} & -d_3^{-1} & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

If now we consider a homogeneous system corresponding to (A.3)

$$\frac{d\mathbf{x}}{dt} = B(t) \mathbf{x} \tag{A.12}$$

and let $\mathbf{y} = D\mathbf{x}$, then we obtain a tridiagonal matrix $DB(t)D^{-1}$ instead of the matrix $B(t)$

$$DB(t)D^{-1} = \begin{pmatrix} -(\lambda_0(t) + \mu_1(t)) & d_0 \times d_1^{-1} \times \mu_1(t) & 0 & \cdots \\ d_1 \times d_0^{-1} \times \lambda_1(t) & -(\lambda_1(t) + \mu_2(t)) & d_1 \times d_2^{-1} \times \mu_2(t) & 0 & \cdots \\ 0 & d_2 \times d_1^{-1} \times \lambda_2(t) - (\lambda_2(t) + \mu_3(t)) & d_2 \times d_3^{-1} \times \mu_3(t) & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with nonnegative extradiagonal elements under all $t \geq 0$.

All further results are obtained with the use of the so-called logarithmic norm of the operator introduced in the finite-dimensional case in [29] and generalized for the Banach space in [30].

The logarithmic norm of the operator function prescribed by the matrix $B(t) = \{b_{ij}(t)\}$ in the space ℓ_1 , is computed by the formula

$$\gamma(B) = \sup_j \left(b_{jj} + \sum_{i \neq j} |b_{ij}| \right). \tag{A.13}$$

For the Cauchy operator $V(t, \tau)$ of the corresponding differential Eq. (A.3) is true the inequality

$$\|V(t, \tau)\| \leq \exp \left(\int_{\tau}^t \gamma(B(s)) ds \right), \quad t \geq \tau \geq 0. \tag{A.14}$$

It is to be noted that if all extradiagonal elements of the matrix $B(t)$ are nonnegative under all $t \geq 0$, then instead of (A.13), we obtain a simpler formula

$$\gamma(B(t)) = \sup_j \sum_i b_{ij}(t). \tag{A.15}$$

Let us now consider quantities (5) when

$$\gamma(B(t))_{1D} = \gamma\left(DB(t)D^{-1}\right)_1 = -\inf_k \alpha_k(t). \tag{A.16}$$

Taking into consideration the triangle inequality for the logarithmic norm and the fact that the logarithmic norm of the operator does not exceed the usual one (i.e., for any operator K under any choice of the norm is true the inequality $\gamma(K) \leq \|K\|$), in any of the considered norms we obtain that

$$\gamma(B^*(t)) \leq \gamma(B(t)) + \|B^*(t) - B(t)\|. \tag{A.17}$$

It is to be noted that

$$\|B^*(t) - B(t)\|_{1D} \leq \sup_k \left((\hat{\lambda}_k(t) + \hat{\mu}_{k+1}(t)) + \frac{d_{k+1}}{d_k} \hat{\lambda}_{k+1}(t) + \frac{d_{k-1}}{d_k} \hat{\mu}_k(t) \right) \leq \Gamma\epsilon. \tag{A.18}$$

Then we can make sure (see [27]) that in this norm

$$\exp\left(\int_{\tau}^t \gamma(B(s))ds\right) = \exp\left(-\int_{\tau}^t \alpha(s)ds\right) \leq M \exp(-\alpha^*(t - \tau)), \tag{A.19}$$

hence, taking into consideration (A.18), we obtain

$$\|U(t, \tau)\|_{1D} \leq \exp\left(\int_{\tau}^t \gamma(B^*(s))ds\right) \leq M \exp((- \alpha^* + \Gamma\epsilon)(t - \tau)). \tag{A.20}$$

Then

$$\int_0^t \|U(t, \tau)\|_{1D} d\tau \leq \frac{M}{\alpha^* - \Gamma\epsilon}. \tag{A.21}$$

Let us now estimate $\|z(\tau)\|_{1D}$. We have

$$\|V(t, \tau)\|_{1D} \leq M e^{-\alpha^*(t-\tau)} \tag{A.22}$$

and

$$\|z(t)\|_{1D} \leq M e^{-\alpha^*t} \|z(0)\|_{1D} + \int_0^t M e^{-\alpha^*(t-\tau)} \|f(\tau)\|_{1D} d\tau \leq M e^{-\alpha^*t} \|z(0)\|_{1D} + \frac{M\alpha\nu_0}{\alpha^*}. \tag{A.23}$$

Hence,

$$\overline{\lim}_{t \rightarrow \infty} \|z(t)\|_{1D} \leq \frac{M\alpha\nu_0}{\alpha^*}. \tag{A.24}$$

Let us now select $z(0)$ so that corresponding $z(t)$ is 1-periodic. Then the mean $\sum_k kp_k(t) = \phi(t)$ corresponding to it will be also 1-periodic and, moreover, according to (A.24) with all t

$$\|z(t)\|_{1D} \leq \frac{M\alpha\nu_0}{\alpha^*}. \tag{A.25}$$

Then selecting $\mathbf{z}^*(0)$ arbitrarily for the time being and taking into consideration that under all $t \geq 0$

$$\|\mathbf{f}(t) - \mathbf{f}^*(t)\|_{1D} = \widehat{\lambda}_0(t) \leq \epsilon, \tag{A.26}$$

we obtain

$$\|\mathbf{z}(t) - \mathbf{z}^*(t)\|_{1D} \leq M e^{(-\alpha^* + \Gamma\epsilon)t} \|\mathbf{z}(0) - \mathbf{z}^*(0)\|_{1D} + \frac{M\epsilon}{\alpha^* - \Gamma\epsilon} \left(1 + \frac{\Gamma M a \nu_0}{\alpha^*}\right). \tag{A.27}$$

If now to take $X^*(0) = k$, then $\|\mathbf{z}^*(0)\|_{1D} = \sum_{i=0}^k d_i$, and then

$$\|\mathbf{z}(t) - \mathbf{z}^*(t)\|_{1D} \leq \|\mathbf{z}(t)\|_{1D} + \|\mathbf{z}^*(t)\|_{1D} \leq \frac{M a \nu_0}{\alpha^*} + \sum_{i=0}^k d_i, \tag{A.28}$$

hence, from (A.27) we obtain

$$\|\mathbf{z}(t) - \mathbf{z}^*(t)\|_{1D} \leq M e^{(-\alpha^* + \Gamma\epsilon)t} \left(\frac{M a \nu_0}{\alpha^*} + \sum_{i=0}^k d_i\right) + \frac{M\epsilon}{\alpha^* - \Gamma\epsilon} \left(1 + \frac{\Gamma M a \nu_0}{\alpha^*}\right). \tag{A.29}$$

Let us now $d_{n-1} = n$ denote the corresponding matrix by E , then the corresponding norm is computed by the formula $\|\mathbf{x}\|_{1E} = \sum n |p_n|$ and taking into consideration the inequality

$$\|\mathbf{x}\|_{1E} \leq (W)^{-1} \|\mathbf{x}\|_{1D} \tag{A.30}$$

we obtain the statement of the theorem.

Proof of Theorem 2. The existence of $\phi^*(t)$ proceeds from estimates (A.20) and (A.30). To obtain estimate (8), it is sufficient to direct t to infinity in inequality (7) under any fixed k .

Proof of Theorem 3. In this case, the BDP state probability vector $X_N(t)$, $t > 0$, complies with the following system instead of system (A.1)

$$\frac{d\mathbf{p}_N}{dt} = A_N(t) \mathbf{p}_N. \tag{A.31}$$

According to Corollary 1 of the paper [27], we obtain estimate

$$|\phi(t) - E \{X_N(t) | X(0) = 0\}| \leq \frac{M^2 a \nu_0}{W \alpha^*} e^{-\alpha^* t} + \frac{3t M a \nu_0 (L a + M b)}{W_N \alpha^*}. \tag{A.32}$$

To complete the proof, it is sufficient to use inequality (7).

Proof of Theorem 4. For the proof, it is sufficient to apply Theorem 2 and also the estimate

$$\left| E - \int_t^{t+1} E \{X_N(u) | X(0) = 0\} du \right| \leq \frac{M^2 a \nu_0}{W \alpha^*} e^{-\alpha^* t} + \frac{3(t+1) M a \nu_0 (L a + M b)}{W_N \alpha^*},$$

obtained in Theorem 3 in [27].

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